Fractional Fokker-Planck Equation
with General Confinement Force

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Fractional Fokker-Planck Equation

We consider the homogeneous **fractional Fokker-Planck Equation**

$$\partial_t f = \Lambda f := \Delta^{\frac{\alpha}{2}} f + \operatorname{div}(E f), \qquad \text{(FFP)}$$

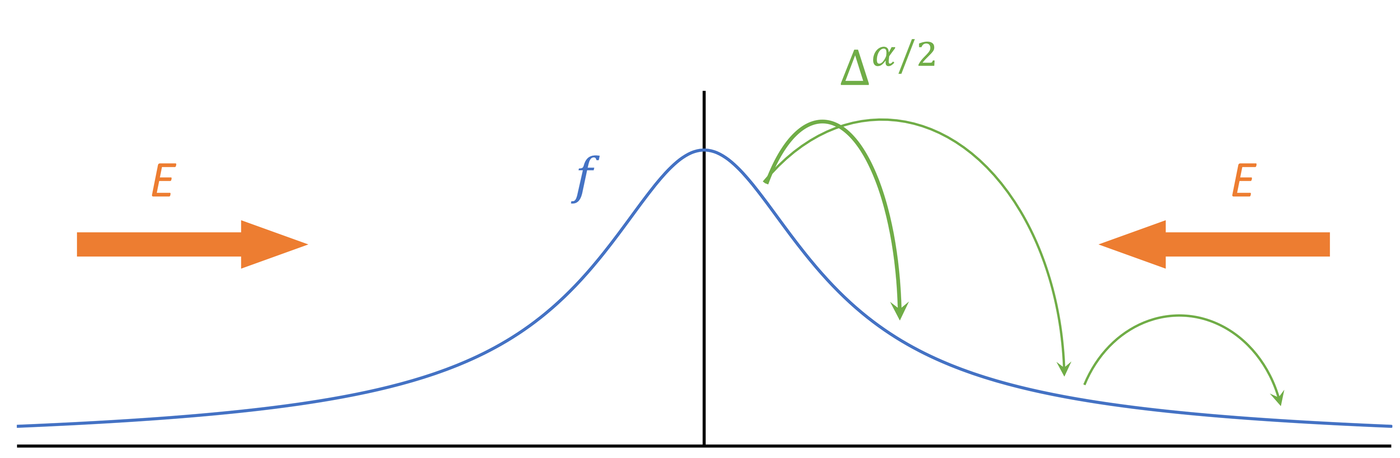
with initial condition $f^{\text{in}} \in L^1$ and where $\Delta^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ is the **fractional Laplacian** defined for example by

$$\Delta^{\frac{\alpha}{2}} u = \mathcal{F}(-|2\pi\xi|^\alpha \hat{u}) \simeq \operatorname{vp}\int_{\mathbb{R}^d} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y.$$

E is a given confining force field taking typically the form

$$E \simeq \langle x\rangle^\beta x \simeq \nabla\left(\langle x\rangle^{2+\beta}\right),$$

where $\langle x\rangle = \sqrt{1+|x|^2}$. We will say that E is **weakly confining** when $\beta \in (-\alpha, 0)$ and that E is **strongly confining** when $\beta \geq 0$. The particular case $E = x$ was studied in [1], [2], [3]) and the classical case $\alpha = 2$ in [4],[5].



Properties of the solutions

The behavior of solutions is studied in weighted Lebesgue spaces defined by

$$\|u\|_{L^p(m)} := \|um\|_{L^p} \qquad \text{with } m(x) = \langle x\rangle^k.$$

where $k < \alpha \wedge 1$. These spaces provide a good framework to understand both the gain of regularity and the gain of space decay. But we still don't know whether or not the solutions become bounded when the force field is strongly confining. However, a special feature of the fractional Laplacian is its nonlocal behavior, which gives a gain of uniform positivity. We end up with the following results:

Proposition 1

Gain of regularity:

$$f^{\text{in}} \in L^1(m) \implies f^{p/2} \in L^1(\mathbb{R}_+, H^{\alpha/2})$$

Gain of integrability:

$$e^{t\Lambda} : L^1(m) \to L^\infty(m) \qquad \text{if } E \text{ is weakly confining.}$$

$$e^{t\Lambda} : L^1 \to L^{p^-_{k,\beta}}(m) \qquad \text{if } E \text{ is strongly confining.}$$

Gain of positivity: for any $N > d + \alpha + \beta$ and $R > 0$, there exists $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$$e^{t\Lambda} f \geq \frac{\psi(t)}{\langle x\rangle^N}\int_{B_R} f$$

Ideas of proof: The gain of regularity and integrability are a consequence of Nash type inequalities coupled with the following estimate for $p < p_{k,\beta}$

$$\partial_t\left(\int_{\mathbb{R}^d} f^p m^p\right) \leq -C\left|(fm)^{\frac{p}{2}}\right|_{H^{\alpha/2}} + \int_{\mathbb{R}^d} |f|^p m^p\left(\frac{c}{\langle x\rangle^\alpha} + b - a\langle x\rangle^\beta\right). \qquad (1)$$

To prove the gain of positivity, we isolate the nonlocal part of the fractional Laplacian by taking $\chi = \mathbb{1}_{B_r}$ and $\kappa^c = (1-\chi)|x|^{-(d+\alpha)}$ and by writing

$$\Delta^{\frac{\alpha}{2}} u = \kappa^c * u - \|\kappa^c\|_{L^1} u + \int_{|x-y|<r} \frac{(u(y)-u(x))}{|x-y|^{d+\alpha}}\,\mathrm{d}y.$$

The operator Λ is then decomposed into $\Lambda = A + \kappa^c * \cdot$ and by Duhamel's formula

$$f(t) = e^{tA} f^{\text{in}} + \int_0^t e^{sA}(\kappa^c * f)(t-s)\,\mathrm{d}s.$$

The convolution with κ^c transforms the local conservation of mass into a quantitative lower bound, which is preserved by e^{sA} thanks to the maximum principle and a positive subsolution.

Polynomial convergence for weak confinement

The positivity and L^p estimates combined with Krein-Rutman theorem give the existence of a steady state F. As in the classical case, the diffusion let appear a dissipation of L^p norm (or dissipation of **generalized entropy**), which can be expressed thanks to the following generalization of $|\nabla|u|^{p/2}|^2$

$$\mathfrak{D}_p(u) \simeq \Delta^{\frac{\alpha}{2}}(|u|^p) - p u^{p-1}\Delta^{\frac{\alpha}{2}}(u).$$

Writing $h = f/F$ for the solution f to the (FFP) equation, it holds

$$\partial_t\left(\int_{\mathbb{R}^d} |h|^p F\right) \leq -\int_{\mathbb{R}^d} \mathfrak{D}_p(h) F.$$

Since F is bounded, we can prove a local **fractional Poincaré inequality**

$$\int_\Omega h^{p-1}\left(h - \int_\Omega hF\right) F \leq C\int_\Omega \mathfrak{D}_p(h) F.$$

We then combine this estimate with (1) to prove the convergence to equilibrium.

Theorem 1

Assume $\beta \in (-\alpha, 0)$. Then there exists $p^* > 1$ such that for any $p < p^*$ and $\bar{k} \leq k$, there exists $a > 0$ such that

$$\|f(t) - F\|_{L^p(\bar{m})} \leq C\langle t\rangle^{-a}\|f^{\text{in}} - F\|_{L^p(m)}$$

Exponential convergence for strong confinement

In the case of strong confinement, it is not known whether or not the solution becomes immediately bounded. However, the rate of convergence to equilibrium is proved following [6] by using the gain of positivity. We change point of view and define the dual (Markov) semigroup $P_t := e^{t\Lambda^*} \in \mathcal{L}(L^\infty(m^{-1}))$. The $L^1(m)$ estimate is then known as the **Foster-Lyapunov criteria** and is expressed as the existence of $\gamma_t \leq 1$ such that,

$$P_t m \leq \gamma_t m + c.$$

The gain of positivity can be rewritten

$$P_t \geq \langle \nu_t, \cdot\rangle \mathbb{1}_{B_R}.$$

It implies exponential convergence to equilibrium in the spirit of Harris' theorem.

Theorem 2

If $\beta \geq 0$ and $k \in (0, \alpha\wedge 1)$, then there exists $p^* > 1$ and $a > 0$ such that for any $p \in [1, p^*)$

$$\|f(t) - F\|_{L^p(m)} \leq Ce^{-at}\|f^{\text{in}} - F\|_{L^p(m)}$$

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